J. Math. Anal. Appl. 365 (2010) 109–114

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability of standing waves for the Klein–Gordon–Schrödinger system

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ARTICLE INFO

Article history:
Received 3 August 2009
Available online 12 October 2009
Submitted by J. Wei

Keywords:
Klein–Gordon–Schrödinger
Standing waves
Stability

ABSTRACT

We study the orbital stability of standing wave solutions for the Klein–Gordon–Schrödinger system in three space dimensions. It is proved that the standing wave is stable if the frequency is sufficiently large.

1. Introduction

We consider the Klein–Gordon–Schrödinger system with Yukawa coupling in three space dimensions:

$$\begin{cases} i\partial_t u + \Delta u = -2uv, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ \partial_t^2 v - \Delta v + m^2 v = |u|^2, & (t,x) \in \mathbb{R}\times\mathbb{R}^3. \end{cases} \tag{1}$$

The system (1) describes a classical model of the Yukawa interaction of conserved complex nucleon field with neutral real meson field (see [8,22]), where $u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is a complex scalar nucleon field, $v : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is a real scalar meson field, and a positive constant m is the mass of meson. We study the orbital stability of standing wave solutions

$$\big(u(t,x), v(t,x)\big) = \big(e^{i\omega t}\varphi_\omega(x), \psi_\omega(x)\big)$$

of (1), where $\omega > 0$ and a pair of functions $(\varphi_\omega, \psi_\omega)$ is a positive and radial ground state of the stationary problem:

$$\begin{cases} -\Delta\varphi + \omega\varphi = 2\varphi\psi, & x \in \mathbb{R}^3, \\ -\Delta\psi + m^2\psi = |\varphi|^2, & x \in \mathbb{R}^3. \end{cases} \tag{2}$$

In other words, φ_ω is a positive and radial ground state of a scalar equation with nonlocal interaction:

$$-\Delta\varphi + \omega\varphi - 2\big(W_m * |\varphi|^2\big)\varphi = 0, \quad x \in \mathbb{R}^3, \tag{3}$$

and $\psi_\omega = (-\Delta + m^2)^{-1}|\varphi_\omega|^2 = W_m * |\varphi_\omega|^2$, where

$$W_m(x) = \frac{e^{-m|x|}}{4\pi|x|}$$

is the Yukawa potential. The existence of a positive and radial ground state can be proved by the standard variational method (see, e.g., [13]).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.024

It is known that the Cauchy problem for (1) is globally well-posed in the energy space $X = H^1(\mathbb{R}^3,\mathbb{C}) \times H^1(\mathbb{R}^3,\mathbb{R}) \times L^2(\mathbb{R}^3,\mathbb{R})$ (see [2] and also [1,3,8,10,11,19]). That is, for any $(u_0, v_0, w_0) \in X$, there exists a unique global solution $(u, v, \partial_t v) \in C(\mathbb{R}, X)$ of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, w_0)$. Moreover, the solution satisfies the conservation laws:

$$\mathbf{E}\big(u(t), v(t)\big) + \big\|\partial_t v(t)\big\|_{L^2}^2 = \mathbf{E}(u_0, v_0) + \|w_0\|_{L^2}^2, \qquad \big\|u(t)\big\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad t \in \mathbb{R},$$

where

$$\mathbf{E}(u, v) = \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + m^2\|v\|_{L^2}^2 - 2\int_{\mathbb{R}^3} |u|^2 v\,dx. \tag{4}$$

Then the stability of standing waves is defined as follows:

Definition. We say that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, v_0, w_0) \in X$ and $\|(u_0, v_0, w_0) - (\varphi, \psi, 0)\|_X < \delta$, then the solution $(u(t), v(t))$ of (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, w_0)$ satisfies

$$\inf_{\theta\in\mathbb{R},\, y\in\mathbb{R}^3} \big\|\big(u(t), v(t), \partial_t v(t)\big) - \big(e^{i\theta}\varphi_\omega(\cdot + y), \psi_\omega(\cdot + y), 0\big)\big\|_X < \varepsilon$$

for all $t \geqslant 0$. Otherwise, $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is said to be *orbitally unstable*.

For the stability of standing waves of (1), only a partial result has been obtained by [18] previously. We now state our main result in this paper.

Theorem 1. *Let $m > 0$ and $\omega > 0$, and let $(\varphi_\omega, \psi_\omega)$ be a positive and radial ground state of (2). Then there exists a positive constant $\omega_* = \omega_*(m)$ such that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally stable for any $\omega \in (\omega_*, \infty)$.*

Remark. In [13], we proved that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally unstable for sufficiently small $\omega > 0$.

For the massless case $m = 0$, it is proved in [18] that the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally stable for all $\omega > 0$ by using the variational method introduced by Cazenave and Lions [4]. The method used in [18] is partially applicable to the massive case $m > 0$. However, since (1) is not scale invariant in the massive case $m > 0$, it was not clear for which ω standing wave $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is stable. For this reason, little was known for the massive case $m > 0$. Note that by the scaling

$$\varphi_\omega(x) = \omega\tilde{\varphi}_\omega\big(\omega^{1/2}x\big), \qquad \psi_\omega(x) = \omega\tilde{\psi}_\omega\big(\omega^{1/2}x\big), \tag{5}$$

$(\tilde{\varphi}_\omega, \tilde{\psi}_\omega)$ satisfies

$$\begin{cases} -\Delta\varphi + \varphi = 2\varphi\psi, & x \in \mathbb{R}^3, \\ -\Delta\psi + \omega^{-1}m^2\psi = |\varphi|^2, & x \in \mathbb{R}^3. \end{cases} \tag{6}$$

From (5) and (6), we see that Theorem 1 is equivalent to the following.

Theorem 2. *Let $m > 0$ and let Φ_m be a positive and radial ground state of (3) with $\omega = 1$, and $\Psi_m = W_m * |\Phi_m|^2$. Then there exists a positive constant m_* such that the standing wave solution $(e^{it}\Phi_m, \Psi_m)$ of (1) is orbitally stable for any $m \in (0, m_*)$.*

For simplicity of notation, we prove Theorem 2 instead of Theorem 1. Otherwise, we would have to use the tilde $\tilde{}$ to denote rescaled functions many times for the proof of Theorem 1. To prove Theorem 2, we obtain the coerciveness of the linearized operator on the orthogonal complement of $\Phi_m, i\Phi_m$ and $\nabla\Phi_m$ (see Proposition 3 below). Then we show that the coerciveness is a sufficient condition for stability. However, it is difficult to obtain the coerciveness directly. Therefore, we first study a limiting equation. We show that Φ_m converges to a unique positive and radial ground state Φ_0 as $m \to 0$. Using the convergence property, we obtain the coerciveness for sufficiently small $m > 0$.

The plan of this paper is as follows. In Section 2, using the coerciveness of the linearized operator, we give the proof of Theorem 2. The coerciveness is proved in Section 3. The key fact for the proof is that the kernel of the linearized operator of the limiting equation is spanned by only trivial one, which is proved by Wei and Winter [20] (see Proposition 6 below).

Remark. This is the revised version of the unpublished paper [14]. We rephrased the title and rewrote the content. After the paper [14] has been completed, it was proved in [15] that the standing wave solution $(e^{im^2t}\sqrt{2}w_m, w_m)$ of (1) is orbitally stable, where w_m is the unique positive radial solution of

$$-\Delta w + m^2 w - 2w^2 = 0, \quad x \in \mathbb{R}^3.$$

Moreover, it was shown in [16] that in two space dimensions, the standing wave solution $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of (1) is orbitally stable if the frequency ω is sufficiently small.

2. Proof of Theorem 2

In this section, we prove Theorem 2. For $m \geqslant 0$, let Φ_m be a positive and radial ground state of

$$-\Delta\varphi + \varphi - 2\big(W_m * |\varphi|^2\big)\varphi = 0, \quad x \in \mathbb{R}^3. \tag{7}$$

We put

$$G_m(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W_m(x-y)\big|u(x)\big|^2\big|u(y)\big|^2\,dx\,dy,$$

$$S_m(u) = \|u\|_{H^1}^2 - G_m(u),$$

$$E_m(u) = \|\nabla u\|_{L^2}^2 - G_m(u).$$

Note that

$$\mathbf{E}(u,v) = E_m(u) + \big\|\big(-\Delta + m^2\big)^{1/2}\big(v - W_m * |u|^2\big)\big\|_{L^2}^2 \tag{8}$$

for $(u,v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, where $\mathbf{E}$ is defined by (4). For each $u \in H^1(\mathbb{R}^3,\mathbb{C})$, we have

$$\big\langle S_m''(\Phi_m)u, u\big\rangle = 2\langle L_m \Re u, \Re u\rangle + 2\langle M_m \Im u, \Im u\rangle, \tag{9}$$

where

$$\langle M_m v, v\rangle = \|v\|_{H^1}^2 - 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W_m(x-y)\Phi_m(x)^2 v(y)^2\,dx\,dy,$$

$$\langle L_m v, v\rangle = \langle M_m v, v\rangle - 4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W_m(x-y)\Phi_m(x)v(x)\Phi_m(y)v(y)\,dx\,dy.$$

Then we have the following.

Proposition 3. *Let $m \geqslant 0$ and Φ_m be a positive and radial ground state of (7). Then there exists a positive constant m_* such that for any $m \in (0, m_*)$ there exists $\delta_m > 0$ such that*

$$\big\langle S_m''(\Phi_m)u, u\big\rangle \geqslant \delta_m \|u\|_{H^1}^2$$

for all $u \in H^1(\mathbb{R}^3,\mathbb{C})$ satisfying $\Re(u,\Phi_m)_{L^2} = \Re(u, i\Phi_m)_{L^2} = 0$ and $\Re(u,\nabla\Phi_m)_{L^2} = 0$.

We prove Proposition 3 in the next section. The following lemma follows from Proposition 3 (see Grillakis, Shatah, and Strauss [9, Theorem 3.4]).

Lemma 4. *Let $m \in (0, m_*)$ and Φ_m be a positive and radial ground state of (7), where the positive constant m_* is given by Proposition 3. Then there exist positive constants C and ϵ such that*

$$E_m(u) - E_m(\Phi_m) \geqslant C\,\mathrm{dis}(u,\Phi_m)^2 \tag{10}$$

for all $u \in H^1(\mathbb{R}^3)$ satisfying $\mathrm{dis}(u,\Phi_m) < \epsilon$ and $\|u\|_{L^2} = \|\Phi_m\|_{L^2}$, where

$$\mathrm{dis}(u,\Phi) = \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^3} \big\|u - e^{i\theta}\Phi(\cdot + y)\big\|_{H^1}.$$

Now, we are in a position to prove Theorem 2.

Proof of Theorem 2. Suppose that the standing wave $(e^{it}\Phi_m, \Psi_m)$ were not orbitally stable. Then there exist $\epsilon_0 > 0$, a sequence of solutions $\{(u_j(t), v_j(t))\}$ of (1) and $\{t_j\} \subset (0,\infty)$ such that

$$\lim_{j\to\infty}\big\|\big(u_j(0), v_j(0), \partial_t v_j(0)\big) - (\Phi_m, \Psi_m, 0)\big\|_X = 0, \tag{11}$$

$$\rho_j := \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^3}\big\|\big(u_j(t_j), v_j(t_j), \partial_t v_j(t_j)\big) - \big(e^{i\theta}\Phi_m(\cdot + y), \Psi_m(\cdot + y), 0\big)\big\|_X = \epsilon_0. \tag{12}$$

By the conservation laws, we have

$$\mathbf{E}\big(u_j(t_j), v_j(t_j)\big) + \big\|\partial_t v_j(t_j)\big\|_{L^2}^2 = \mathbf{E}\big(u_j(0), v_j(0)\big) + \big\|\partial_t v_j(0)\big\|_{L^2}^2 \to \mathbf{E}(\Phi_m, \Psi_m) = E_m(\Phi_m),$$

$$\big\|u_j(t_j)\big\|_{L^2}^2 = \big\|u_j(0)\big\|_{L^2}^2 \to \|\Phi_m\|_{L^2}^2$$

as $j \to \infty$. Here we put $\alpha_j = \|\Phi_m\|_{L^2}/\|u_j(t_j)\|_{L^2}$. Then, $\alpha_j \to 1$, and by (10), we have

$$E_m(\Phi_m) \leqslant E_m\big(\alpha_j u_j(t_j)\big) \leqslant \mathbf{E}\big(\alpha_j u_j(t_j), v_j(t_j)\big) + \big\|\partial_t u_j(t_j)\big\|_{L^2}^2 \to E_m(\Phi_m).$$

Therefore, by (8), we have

$$E_m\big(u_j(t_j)\big) \to E_m(\Phi_m), \qquad \big\|v_j(t_j) - W_m * \big|u_j(t_j)\big|^2\big\|_{H^1} \to 0, \qquad \big\|\partial_t v_j(t_j)\big\|_{L^2} \to 0.$$

Again by (10), we see that $\operatorname{dis}(u_j(t_j), \Phi_m) \to 0$, and we conclude that $\rho_j \to 0$. This contradicts (12). □

3. Proof of Proposition 3

In this section, we prove the following Proposition 5. By (9), Proposition 3 follows from Proposition 5.

Proposition 5. *Let $m > 0$ and Φ_m be a positive and radial ground state of* (7)*. Then there exists a positive constant m_* such that for any $m \in (0, m_*)$ there exists $\delta_m > 0$ such that*

(i) $\langle L_m v, v\rangle \geqslant \delta_m \|v\|_{H^1}^2$ *for all $v \in H^1(\mathbb{R}^3, \mathbb{R})$ satisfying $(v, \Phi_m)_{L^2} = 0$, $(v, \nabla\Phi_m)_{L^2} = 0$.*
(ii) $\langle M_m v, v\rangle \geqslant \delta_m \|v\|_{H^1}^2$ *for all $v \in H^1(\mathbb{R}^3, \mathbb{R})$ satisfying $(v, \Phi_m)_{L^2} = 0$.*

To prove Proposition 5, we use the variational characterization of ground state Φ_m. Note that Φ_m is a minimizer of the following minimization problem:

$$d_m = \inf\big\{S_m(u)\colon\ u \in H^1\big(\mathbb{R}^3, \mathbb{R}\big) \setminus \{0\},\ K_m(u) = 0\big\},$$

where $K_m(u) = \|u\|_{H^1}^2 - 2G_m(u)$.

We first consider the massless case $m = 0$. The uniqueness of positive radial solution $\Phi_0 \in H^1(\mathbb{R}^3)$ of (7) is proved by Lieb [12] (see also Choquard, Stubbe and Vuffray [5] for more general result). The following Proposition 6 is proved by Wei and Winter [20, Theorem 3.1].

Proposition 6. $\ker L_0 = \operatorname{span}\{\partial_1\Phi_0, \partial_2\Phi_0, \partial_3\Phi_0\}$.

Lemma 7. $\inf\{\langle S_0''(\Phi_0)v, v\rangle\colon\ v \in H^1(\mathbb{R}^3, \mathbb{C}), \Re(v, \Phi_0)_{L^2} = 0\} = 0$. *In particular,*

$$\tau := \inf\big\{\langle L_0 w, w\rangle\colon\ w \in H^1\big(\mathbb{R}^3, \mathbb{R}\big), (w, \Phi_0)_{L^2} = 0\big\} = 0.$$

Proof. Our proof borrows some elements from Lemma 2.2 of Maris [17]. It is clear that $\tau \leqslant 0$. Let $v \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfy $\Re(v, \Phi_0)_{L^2} = 0$. Then, we have $\|t\Phi_0 + sv\|_{L^2}^2 = t^2\|\Phi_0\|_{L^2}^2 + s^2\|v\|_{L^2}^2$ for $s, t \in \mathbb{R}$. Here we define

$$t(s) := \left(1 - s^2\frac{\|v\|_{L^2}^2}{\|\Phi_0\|_{L^2}^2}\right)^{1/2}, \qquad w(s) := t(s)\Phi_0 + sv$$

for s close to 0. Then, we have $\|w(s)\|_{L^2}^2 = \|\Phi_0\|_{L^2}^2$, $w(0) = \Phi_0$ and $w'(0) = v$. Since Φ_0 satisfies

$$E_0(\Phi_0) = \min\big\{E_0(u)\colon\ u \in H^1\big(\mathbb{R}^3, \mathbb{C}\big), \|u\|_{L^2} = \|\Phi_0\|_{L^2}\big\}$$

(see [12]), the function $s \mapsto E_0(w(s))$ achieves a local minimum at $s = 0$, we have

$$\begin{aligned} 0 &\leqslant \frac{d^2}{ds^2}E_0\big(w(s)\big)\Big|_{s=0} = \frac{d^2}{ds^2}S_0\big(w(s)\big)\Big|_{s=0} \\ &= \big\langle S_0'\big(w(0)\big), w''(0)\big\rangle + \big\langle S_0''\big(w(0)\big)w'(0), w'(0)\big\rangle = \big\langle S_0''(\Phi_0)v, v\big\rangle. \end{aligned}$$

This completes the proof. □

Next, we show the following convergence property.

Proposition 8. *Let $m \geqslant 0$ and let Φ_m be a positive and radial ground state of* (7)*. Then,*

(i) $\lim_{m\to+0}\|\Phi_m\|_{H^1}^2 = \|\Phi_0\|_{H^1}^2$.
(ii) $\lim_{m\to+0} G_0(\Phi_m) = G_0(\Phi_0)$.
(iii) *$\Phi_m \to \Phi_0$ strongly in $H^1(\mathbb{R}^3)$ as $m \to +0$.*

Proof. (i) First, by a similar argument in [13, Lemma 1], we see that

$$4d_m = \|\Phi_m\|_{H^1}^2 = \inf\{\|u\|_{H^1}^2 \colon u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\}, K_m(u) \leqslant 0\}. \tag{13}$$

Since $K_0(\Phi_m) \leqslant K_m(\Phi_m) = 0$, we have $\|\Phi_0\|_{H^1}^2 \leqslant \|\Phi_m\|_{H^1}^2$ for all $m > 0$. Next, for each $\mu > 1$, we have

$$K_m(\mu\Phi_0) = \mu^2\|\Phi_0\|_{H^1}^2 - 2\mu^4 G_m(\Phi_0) = 2\mu^2 G_0(\Phi_0) - 2\mu^4 G_m(\Phi_0).$$

By the dominated convergence theorem, we have $G_m(\Phi_0) \to G_0(\Phi_0)$ as $m \to +0$. Thus, there exists $m(\mu) > 0$ such that $K_m(\mu\Phi_0) < 0$ for all $m \in (0, m(\mu))$. Therefore, for each $\mu > 1$, we have $\|\Phi_0\|_{H^1}^2 \leqslant \|\Phi_m\|_{H^1}^2 \leqslant \mu^2\|\Phi_0\|_{H^1}^2$ for all $m \in (0, m(\mu))$, which shows (i).

(ii) By (i), we have $K_0(\Phi_m) \leqslant K_m(\Phi_m) = 0$ for all $m > 0$. Next, we define a positive constant μ_m by

$$\mu_m^2 = \frac{\|\Phi_m\|_{H^1}^2}{2G_0(\Phi_m)}.$$

Then we have $K_0(\mu_m\Phi_m) = 0$. By (13) we have $\|\Phi_0\|_{H^1}^2 \leqslant \mu_m^2\|\Phi_m\|_{H^1}^2$. Moreover, since $K_0(\Phi_m) \leqslant 0$, we have

$$\frac{\|\Phi_0\|_{H^1}^2}{\|\Phi_m\|_{H^1}^2} \leqslant \mu_m^2 = \frac{\|\Phi_m\|_{H^1}^2}{2G_0(\Phi_m)} \leqslant 1.$$

By (i), we see that $\lim_{m\to+0}\mu_m = 1$, which proves (ii).

(iii) Finally, by (i) and (ii), we see that $\{\mu_m\Phi_m\}$ is a minimizing sequence for d_0. Since $\{\mu_m\Phi_m\}$ has a convergent subsequence and Φ_0 is the unique minimizer for d_0, we obtain (iii). □

The following lemma is used in the proof of Proposition 5.

Lemma 9. *Let $\{m_j\} \subset (0,\infty)$ with $m_j \to 0$ and let $\{v_j\} \subset H^1(\mathbb{R}^3)$ and $w \in H^1(\mathbb{R}^3)$ satisfy $v_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^3)$. Then we have*

$$\lim_{j\to\infty} \mathcal{W}_{m_j}(\Phi_{m_j}^2, v_j^2) = \mathcal{W}_0(\Phi_0^2, w^2), \qquad \lim_{j\to\infty} \mathcal{W}_{m_j}(\Phi_{m_j}v_j, \Phi_{m_j}v_j) = \mathcal{W}_0(\Phi_0 w, \Phi_0 w),$$

where we put

$$\mathcal{W}_m(f,g) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W_m(x-y)f(x)g(y)\,dx\,dy = \int_{\mathbb{R}^3} g(x)(W_m * f)(x)\,dx.$$

Proof. We first remark that by the Hardy–Littlewood–Sobolev inequality, $\|W_m * f\|_{L^6} \leqslant C\|f\|_{L^{6/5}}$ for all $f \in L^{6/5}(\mathbb{R}^3)$ and for all $m \geqslant 0$, where the constant C is independent of m. Moreover, since $\Phi_{m_j} \to \Phi_0$ strongly in $H^1(\mathbb{R}^3)$ and $v_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^3)$, we see that $\Phi_{m_j}^2 \to \Phi_0^2$ strongly in $L^{6/5}(\mathbb{R}^3)$, $v_j^2 \rightharpoonup w^2$ weakly in $L^{6/5}(\mathbb{R}^3)$ and $\Phi_{m_j}v_j \to \Phi_0 w$ strongly in $L^{6/5}(\mathbb{R}^3)$. In particular, $W_{m_j} * \Phi_{m_j}^2 \to W_0 * \Phi_0^2$ strongly in $L^6(\mathbb{R}^3)$ and $W_{m_j} * (\Phi_{m_j}v_j) \to W_0 * (\Phi_0 w)$ strongly in $L^6(\mathbb{R}^3)$. Thus, we have

$$\mathcal{W}_{m_j}(\Phi_{m_j}^2, v_j^2) = \int_{\mathbb{R}^3} v_j^2(x) W_{m_j} * \Phi_{m_j}^2(x)\,dx \to \int_{\mathbb{R}^3} w^2(x) W_0 * \Phi_0^2(x)\,dx = \mathcal{W}_0(\Phi_0^2, w^2)$$

and $\mathcal{W}_{m_j}(\Phi_{m_j}v_j, \Phi_{m_j}v_j) \to \mathcal{W}_0(\Phi_0 w, \Phi_0 w)$. □

Following Esteban and Strauss [7], Weinstein [21] and de Bouard and Fukuizumi [6], we prove Proposition 5.

Proof of Proposition 5. We prove (i) only. The proof of (ii) is similar and easier. Suppose that (i) were false. Then there exist $\{m_j\} \subset (0,\infty)$ and $\{v_j\} \subset H^1(\mathbb{R}^3, \mathbb{R})$ such that $m_j \to +0$ and

$$\liminf_{j\to\infty}\langle L_{m_j}v_j, v_j\rangle \leqslant 0, \qquad \|v_j\|_{H^1} = 1, \qquad (v_j, \Phi_{m_j})_{L^2} = 0, \qquad (v_j, \nabla\Phi_{m_j})_{L^2} = 0.$$

Since $\{v_j\}$ is bounded in $H^1(\mathbb{R}^3)$, there exists a subsequence of $\{v_j\}$ (we still denote it by $\{v_j\}$) and $w \in H^1(\mathbb{R}^3)$ such that $v_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^3)$. Then, by Lemma 9, we have

$$0 \geqslant \liminf_{j\to\infty}\langle L_{m_j}v_j, v_j\rangle = 1 - 2\mathcal{W}_0(\Phi_0^2, w^2) - 4\mathcal{W}_0(\Phi_0 w, \Phi_0 w),$$

which implies $w \neq 0$. On the other hand, we have

$$\langle L_0 w, w\rangle \leqslant \liminf_{j\to\infty}\langle L_{m_j}v_j, v_j\rangle \leqslant 0.$$

Moreover, since $v_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^3)$ and $\Phi_{m_j} \to \Phi_0$ strongly in $H^1(\mathbb{R}^3)$, we have

$$(w, \Phi_0)_{L^2} = \lim_{j\to\infty} (v_j, \Phi_{m_j})_{L^2} = 0, \qquad (w, \nabla\Phi_0)_{L^2} = \lim_{j\to\infty} (v_j, \nabla\Phi_{m_j})_{L^2} = 0.$$

By Lemma 7, w is a minimizer for τ, so there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $L_0 w = \lambda\Phi_0$. Let

$$\phi(x) = 2\Phi_0(x) + x \cdot \nabla\Phi_0(x).$$

Then we have $L_0\phi = \Phi_0$, so $w - \lambda\phi \in \ker L_0$. By Proposition 6, there exists $\alpha \in \mathbb{R}^3$ such that $w = \lambda\phi + \alpha \cdot \nabla\Phi_0$. Then we have

$$0 = (w, \Phi_0)_{L^2} = \lambda(\phi, \Phi_0)_{L^2} = \frac{\lambda}{2}\|\Phi_0\|_{L^2}^2,$$

so $\lambda = 0$. Thus, we have $w = \alpha \cdot \nabla\Phi_0$. Since $(w, \nabla\Phi_0)_{L^2} = 0$, we have $w = 0$. This is a contradiction. Hence, we obtain (i). □

Acknowledgments

The first author would like to express his deep gratitude to Professor Yoshio Tsutsumi for his valuable comments. The research of the first author was partly supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. The research of the second author was partly supported by KAKENHI (18540161).

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